Adiabatic Compression

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Introduction

Let's say we have some gas trapped inside a cylinder with a piston preventing it from leaking. Now if we want to change the inner energy of the gas we can - according to the first law of Thermodynamics - either add/remove heat from the system or do work on the gas or let the gas do work on the environment.

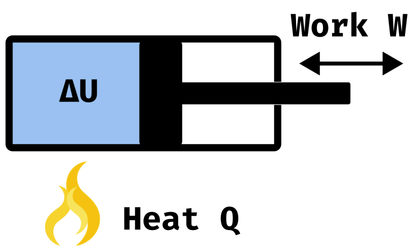

Figure 1: Piston

In a formula you can write the first law of Thermodynamics like this

> **!** First law of Thermodynamics
>
> $$dU = dQ + dW$$
>
> where dU is the change of the inner energy of the gas, dQ is the amount of exchanged heat and dW is the amount of work being done on the gas/ on the environment.

Adiabatic process

During an adiabatic process there is by definition no heat exchanged with the environment. This can e.g. be approximately true for a very fast change in volume where there is almost no time to allow heat flow during this process. For us this is very convenient as we can now write the first law like this

$$dU = dW$$

The inner energy U for an ideal gas is defined by

$$U = \frac{f}{2} N k_B T$$

where f is the degree of freedom for the gas, N the number of gas particles, k_B the Boltzmann constant and T the temperature of the gas. As T is the only variable that might change during the adiabatic process we can write the change of inner energy as

$$dU = \frac{f}{2} N k_B dT$$

The work done on the gas/ on the environment is given by

$$dW = -pdV$$

where p is the pressure and dV the change of volume during the process.

Combining those to the first equation we get

$$\frac{f}{2} N k_B dT = -pdV$$

Now let's introduce probably the most essential identity of Thermodynamics

> **! The Ideal Gas Law**
>
> $$pV = N k_B T$$
>
> where p is the pressure, V the volume, N the number of gas particles, k_B the Boltzmann constant and T the temperature of the gas.

We can now apply the differential operator to get $dpV + pdV = N k_B dT$ and insert this into our equation

$$\frac{f}{2}(V dp + pdV) = -pdV$$

$$\left(\frac{f}{2} + 1\right) pdV = -\frac{f}{2} V dp$$

$$\frac{f+2}{f}\frac{1}{V}dV = -\frac{1}{p}dp$$

where one defines $\gamma = \frac{f+2}{f}$ as the adiabatic index.

Let's integrate this equation on both sides

$$\gamma \ln(V) = -\ln(p) + \text{const}$$

Using $a\ln(x) = \ln(x^a)$ and $\ln(x) + \ln(y) = \ln(xy)$ we get

! Adiabatic Relation for Pressure and Volume

$$pV^\gamma = \text{const}$$

where p is the pressure, V the volume and γ the adiabatic index of the gas.

Now by inserting $p = \frac{1}{V}Nk_BT$ we get

$$Nk_BTV^{-1}V^\gamma = \text{const}$$

! Adiabatic Relation for Temperature and Volume

$$TV^{\gamma-1} = \text{const}$$

where T is the temperature, V the volume and γ the adiabatic index of the gas.

Example

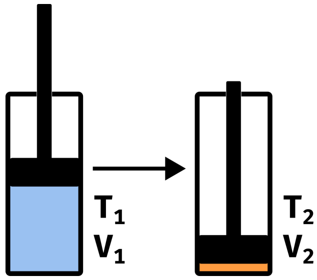

Figure 2: Adiabatic Heating

As you may know, the Diesel Engine does not rely on spark ignition - the fuel can ignite without a spark needed. The fuel-air vapor is instead compressed so quickly that the temperature increases significantly - enough to start the power stroke.

E.g. The 2019 Ford Super Duty has a compression ratio of 16.2 : 1. We derived that $TV^{\gamma-1} = \text{const}$ and so

$$T_1 V_1^{\gamma-1} = T_2 V_2^{\gamma-1}$$

where T_1 is the temperature before the compression and T_2 the temperature after the compression (same for volume). Let's assume that the incoming air is about $T_1 = 300K$ and that we deal only with a diatomic gas with $f = 5$ and therefore $\gamma = \frac{7}{5}$.

The final temperature can then be calculated with

$$T_2 = \left(\frac{V_1}{V_2}\right)^{\gamma-1} T_1$$

Plugging in our values we get

$$T_2 = \left(\frac{16.2}{1}\right)^{\frac{2}{5}} 300K \approx 914K = 641°C$$

Sources

- Inkscape Tutorial: Vector Flame Icon
- 2019 Ford Super Duty Specifications